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On AX and ABX Limens

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Rosenblith and Stevens obtained a larger frequency limen by the ABX method than by the AX method. The ratio of the limens is here calculated on the assumption that there is a statistical variation in the physiological or psychological response to a given stimulus. The ratio of limens is 2.12 for a Gaussian distribution of response and 2.14 for a uniform distribution. The model fits data for subject KNS well, but not the data for subject WAR.

ROSENBLITH and Stevens[1] obtained a larger limen for frequency by the ABX method than they did by the AX method. Here a statistical explanation of this difference is proposed.

Let us assume that in the AX comparison the frequency of X is actually larger than that of A. We will assume, however, that there is a random uncertainty in the physiological or psychological response to an acoustic stimulus of a given frequency. We will assume that the probability $p(u)du$ of obtaining a response to the stimulus A of magnitude u lying in the range du is

$$p(u)du=\frac{1}{\sigma(2\pi)^{\frac{1}{2}}}\exp\left[-\frac{1}{2}\left(\frac{u}{\sigma}\right)^2\right]du, \tag{1}$$

and that the probability $p(v)dv$ of obtaining a response to the stimulus X of magnitude v lying in the range dv is

$$p(v)dv=\frac{1}{\sigma(2\pi)^{\frac{1}{2}}}\exp\left[-\frac{1}{2}\left(\frac{v-d}{\sigma}\right)^2\right]dv. \tag{2}$$

Here d is a difference in mean response which we will assume to be proportional to the difference between the stimuli.

If the organism can correctly judge whether v is greater or less than u, then a correct judgment that X is greater than A will be made when

$$v>u. \tag{3}$$

Using this criterion, the probability that it will correctly infer when X is greater than A is shown in the Appendix to be

$$P_{AX}=\tfrac{1}{2}[1+\mathrm{erf}(d/2\sigma)], \tag{4}$$

where

$$\mathrm{erf}x=\frac{2}{\pi^{\frac{1}{2}}}\int_0^x \exp(-y^2)dy. \tag{5}$$

The probability of a correct judgment will be the same when X is less than A by a like amount.

In the ABX test, X is identical with either A or B; in our mathematical model it does not matter which. In this case, the probabilities that A will evoke a response u, B a response v, and X a response w are taken as

$$p(u)du=\frac{1}{\sigma(2\pi)^{\frac{1}{2}}}\exp\left[-\frac{1}{2}\left(\frac{u}{\sigma}\right)^2\right]du, \tag{6}$$

$$p(v)dv=\frac{1}{\sigma(2\pi)^{\frac{1}{2}}}\exp\left[-\frac{1}{2}\left(\frac{v-d}{\sigma}\right)^2\right]dv, \tag{7}$$

$$p(w)dw=\frac{1}{\sigma(2\pi)^{\frac{1}{2}}}\exp\left[-\frac{1}{2}\left(\frac{w}{\sigma}\right)^2\right]dw. \tag{8}$$

[1] Walter H. Rosenblith and Kenneth N. Stevens, J. Acoust. Soc. Am. 25, 980–985 (1953).

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TABLE I.

Frequency	Sensation level db	KNS $\frac{DL_{ABX}}{DL_{AX}}$	WAR $\frac{DL_{ABX}}{DL_{AX}}$
250	30	2.1	3.1
1000	30	2.1	2.5
1000	60	2.1	3.0
4000	30	1.8	4.1

This implies that stimulus X is identical with stimulus A. The organism will make a correct judgment whenever

$$|u-w|<|v-w|. \tag{9}$$

In the appendix it is shown that according to this criterion, the probability that the organism will make a correct judgment is

$$P_{ABX}=\frac{1}{2}\left[1+\operatorname{erf}\left(\frac{d}{2\sigma}\right)\operatorname{erf}\left(\frac{d}{2\sigma\sqrt{3}}\right)\right]. \tag{10}$$

The limen is defined as the difference between stimuli for which the probability of a correct judgment is 0.75. We have

$$P_{AX}=0.75 \quad \text{when} \quad d/2\sigma=0.477 \tag{11}$$

$$P_{ABX}=0.75 \quad \text{when} \quad d/2\sigma=1.01. \tag{12}$$

Hence, if the mean difference of response d is proportional to the difference between stimuli, the ratio of DL_{ABX}, the limen for ABX test, to DL_{AX}, the limen for the AX test, should be

$$DL_{ABX}/DL_{AX}=1.01/0.477=2.12. \tag{13}$$

Values from Table I of the paper by Rosenblith and Stevens[1] are shown as our Table I.

The model certainly fits the behavior of KNS admirably; it does not explain the large ratio commonly exhibited by WAR.

It should be noted that the ratio DL_{ABX}/DL_{AX} is not highly sensitive to the type of distribution assumed. Thus, if one assumes that u, v, w, have uniform distributions over the interval L, there will be no wrong guesses in either the AX or the ABX case unless $d<L$. However, when $d<L$,

$$\text{Prob (correct guess in AX test)}=1-\frac{1}{2}\left(1-\frac{d}{L}\right)^2 \tag{14}$$

Prob (correct guess in ABX test)

$$=\frac{1}{2}+\left(\frac{d}{L}\right)^2-\frac{7}{12}\left(\frac{d}{L}\right)^3. \tag{15}$$

If we choose (d/L) such as to give a 0.75 probability of a correct guess in Eqs. (14) and (15) we find

$$DL_{ABX}/DL_{AX}=2.14. \tag{16}$$

We see that this is not very different from the value for the Gaussian distribution, given in Eq. (13).

In the foregoing calculations it has been assumed that in the ABX test the observer does not know whether the higher or the lower reference tone will be presented first; that is, the orders of presentation high-low and low-high are chosen randomly and independently with equal probabilities. This was the case in the work of Rosenblith and Stevens.[1]

David M. Green has pointed out to the authors that as originally proposed by Munson and Gardner[2] and as used subsequently by Munson and Karlin,[3] the order of the reference stimuli in the ABX test was invariant; this would correspond to presenting the higher frequency always either first or second. In this case, the subject *knows* the order of presentation of the reference tones; if A is in fact less than B, and if he judges that X is less than the mean of A and B then he must conclude that X is equal to A. As he cannot compare X with A and B, but must compare w with the mean of v and u, the probability of correct judgment is the probability that w is smaller than the mean of v and u. For a Gaussian distribution this turns out to be

$$P_{ABX}=\frac{1}{2}\left[1+\operatorname{erf}\left(\frac{d}{2\sigma\sqrt{3}}\right)\right]. \tag{17}$$

This is equal to 0.75 when

$$d/2\sigma=0.827. \tag{18}$$

Thus, for this sort of ABX test and for a Gaussian distribution we would expect

$$DL_{ABX}/DL_{AX}=1.73. \tag{19}$$

It would be interesting to have experimental data pertaining to the two sorts of tests to which Eqs. (13) and (19) apply.

The remarks of Harris[4] on ABX limens are of interest.

APPENDIX

AX Test

Correct guess requires $v>u$, i.e., $v-u>0$. Since v and u are Gaussian random variables, so is $v-u$. It has mean d and variance $2\sigma^2$ where σ^2 is the variance of v and of u. Then

$$\text{prob}(v>u)=\text{prob}(v-u>0)$$

$$=\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-d/\sigma\sqrt{2}}^{\infty}\exp(-t^2/2)dt.$$

This can be reduced to

$$\text{prob}(v>u)=\tfrac{1}{2}[1+\operatorname{erf}(d/2\sigma)].$$

[2] W. A. Munson and Mark B. Gardner, J. Acoust. Soc. Am. 22, 177–190 (1950).

[3] W. A. Munson and J. E. Karlin, J. Acoust. Soc. Am. 26, 542–553 (1954).

[4] J. Donald Harris, J. Acoust. Soc. Am. 24, 417 (1952).

ABX Test

Correct guess requires

$$|u-w|<|v-w|.$$

Let

$$\xi=u+v-2w-d$$

and

$$\eta=u-v+d.$$

Note that

$$u-w=\tfrac{1}{2}(\xi+\eta)$$

and

$$v-w=\tfrac{1}{2}(\xi-\eta)+d$$

so that condition for correct guess is

$$|\xi+\eta|<|\xi-\eta+2d|,$$

i.e., (ξ,η) is a point in the shaded region of Fig. 1.

The problem is already reduced to a double integration if we can find the joint distribution for ξ, η. Note ξ and η are jointly Gaussian with means 0. The second-order moments are

$$\begin{aligned}E(\xi^2)&=\sigma^2+\sigma^2+4\sigma^2=6\sigma^2\\E(\eta^2)&=\sigma^2+\sigma^2=2\sigma^2\\E(\xi\eta)&=E[(u+v)(u-v)\\&\qquad+(-2w-d)(u-v)+d(u+v-2w-d)]\\&=E(u^2)-E(v^2)\\&\qquad+E(-2w-d)E(u-v)+dE(u+v-2w-d)\\&=\sigma^2-\sigma^2+0+0\\&=0.\end{aligned}$$

Then ξ and η are independent and the probability of a correct guess is just

$$\int\int \frac{\exp(-\xi^2/12\sigma^2)d\xi}{(2\pi)^{\frac{1}{2}}(6)^{\frac{1}{2}}\sigma}\cdot\frac{\exp(-\eta^2/4\sigma^2)d\eta}{(2\pi)^{\frac{1}{2}}(2)^{\frac{1}{2}}\sigma},$$

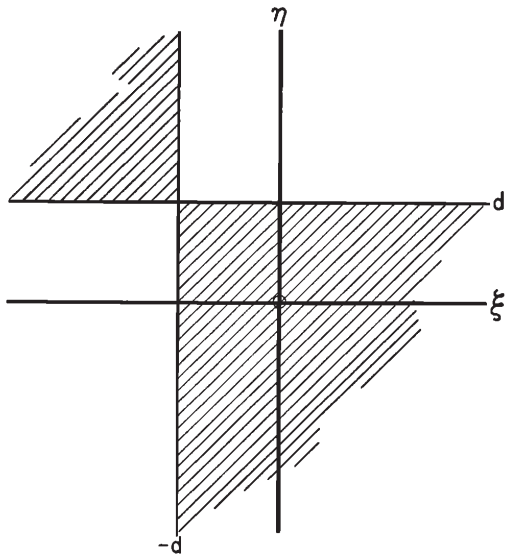

FIG. 1.

where the region of integration is the one shaded in Fig. 1.

Prob(correct guess)

$$=\int_{-\infty}^{-d/\sigma(6)^{\frac{1}{2}}}\frac{\exp(-s^2/2)ds}{(2\pi)^{\frac{1}{2}}}\int_{d/\sigma\sqrt{2}}^{\infty}\frac{\exp(-t^2/2)}{(2\pi)^{\frac{1}{2}}}dt+\int_{-d/\sigma(6)^{\frac{1}{2}}}^{\infty}\frac{\exp(-s^2/2)ds}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{d/\sigma\sqrt{2}}\frac{\exp(-t^2/2)dt}{(2\pi)^{\frac{1}{2}}}.$$

This can be reduced to

$$\text{prob}(|u-w|<|v-w|)=\frac{1}{2}\left[1+\text{erf}\left(\frac{d}{2\sigma}\right)\text{erf}\left(\frac{d}{2\sigma\sqrt{3}}\right)\right].$$